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On derivations of MV-algebras

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Abstract

In this paper, we investigate related properties of some particular derivations and give some characterizations of additive derivations in MV-algebras. Then, we obtain that the fixed point set of additive derivations is still an MV-algebra. Also, we study Boolean additive derivations and their adjoint derivations. In particular, we get that the fixed point set of Boolean addition derivations and that of their adjoint derivations are isomorphism. Moreover, we prove that every MV-algebra is isomorphic to the direct product of the fixed point set of Boolean additive derivations and that of their adjoint derivations. Finally, we show that the structure of a Boolean algebra is completely determined by its set of all Boolean additive (implicative) derivations.

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1. Introduction

It is well known that certain information processing approaches, especially inferences based on certain information, are based on the classical logic (classical two-valued logic). Naturally, it is necessary to establish some rational logic systems as a logical foundation for uncertain information processing. For this reason, various types of non-classical logic systems have been proposed and researched. In recent years, non-classical logic has become a formal and useful tool in computer science for dealing with uncertain and fuzzy information. Various logical algebras have been proposed as semantic systems for non-classical logic systems. Among these logical algebras, MV-algebras are the first class of logic algebras introduced and investigated. In 1958, C.C.Chang introduced the notion of MV-algebras for the purpose of providing algebraic proof of the completeness theorem of infinite-valued propositional logics [8]. We are speaking here of the infinite-valued logic proposed in 1930 by Łukasiewicz and Tarski [18] with truth values in the internal [0,1] of real numbers. Thus, in a certain sense, MV-algebras stand in relations to many-valued

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logic as Boolean algebras do to classical two-valued logic. Furthermore, Chang [9] established a bijective correspondence between the linearly ordered MV-algebras and the linearly ordered abelian ℓ -groups with strong unit, then used this result in order to obtain an algebraic proof for the completeness theorem of Łukasiewicz propositional logic in another way. In [10], Mundici extended Chang's result by proving the categorial equivalence between MV-algebras and abelian ℓ -groups with strong unit.

The notion of derivations, introduced from the analytic theory, is helpful for studying algebraic structures and properties in algebraic systems. In fact, the notion of derivation in ring theory is quite old and plays a significant role in algebraic geometry. In 1957, Posner[19] introduced the notion of derivations in prime rings. After then, a number of research articles have been appeared on derivations in the theory of rings and references there in [5, 6, 15, 1]. Inspired by derivations on rings, Jun et al [17] applied the notion of derivations to BCI-algebras and gave some characterizations of p-semisimple BCI-algebras. Based on this, several authors have studied derivations in BCI-algebras [24, 3, 7]. In 2010, Alshehri[2] applied the notions of (additive) derivations to MV-algebras and discussed some related properties, they also proved that an additive derivation of a linearly ordered MV-algebra is isotone. After the work of Alshehri, many authors have investigated derivations of MV-algebras in different aspects [23, 13, 4], for example, Yazarli et al [23] further investigated several kinds of generalized derivations on MV-algebras and obtain some interesting results. Ardekani and Davvaz [4] introduced the notion of f -derivations and (f, g) -derivations of MV -algebras and investigated some related properties of them. Ghorbain et al [13] introduced several kinds of these derivations and discuss some related results. They also discuss the relationship between these derivations on MV-algebras. Recently, the notion of derivations has been extended to various logical algebras such as BL-algebras [22], residuated lattices [16] and their non-commutative cases.

This paper is a continuation of the research from [2]. The paper is organized as follows: In Section 2, we review some basic definitions and results about MV-algebras. In Section 3, we characterize some particular derivations in MV-algebras. In Section 4, we study Boolean derivations and their adjoint derivations. In particular, we show that every MV-algebra is isomorphic to the direct product of the fixed point set of Boolean additive derivations and that of their adjoint derivations.

2. Preliminaries

In this section, we summarize some definitions and results about MV-algebras, which will be used in the following sections.

Definition 2.1. [8] An algebra $(L, \oplus, *, 0)$ of type $(2, 1, 0)$ is called an *MV-algebra* if it satisfies the following conditions:

- (MV1) $(L, \oplus, 0)$ is a commutative monoid,
- (MV2) $(x^*)^* = x$,
- (MV3) $0^* \oplus x = 0^*$,

(MV4) $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$, for any $x, y \in L$.

In what follows, by L we denote the universe of an MV-algebra $(L, \oplus, *, 0)$. On each MV-algebra L , we define the constant 1 and the operations $\odot, \ominus, \rightarrow$ as follows: $1 = 0^*$, $x \odot y = (x^* \oplus y^*)^*$, $x \ominus y = x^* \odot y$ and $x \rightarrow y = x^* \oplus y$. We define $x \leq y$ if and only if $x^* \oplus y = 1$. It follows that \leq is a partial order, called the *natural order* of L . On each MV-algebra, the natural order determines a lattices structure, in which, $x \vee y = (x \odot y^*) \oplus y$, $x \wedge y = x \odot (x^* \oplus y)$. In fact, one can prove that $(L, \wedge, \vee, 0, 1)$ is a distributive lattice. Since MV-algebras form a variety, the notions of homomorphism, subalgebra are just the particular cases of the corresponding universal algebraic notions [8, 9, 18].

Example 2.2. [11] Let $L = [0, 1]$ be the real unit interval. If we define $x \oplus y = \min\{1, x + y\}$ and $x^* = 1 - x$ for any $x, y \in L$, then $(L, \oplus, *, 0)$ is an MV-algebra. Also, for each number $n \geq 2$, then n -element set $S_n = \{0, \frac{1}{n-1}, \frac{2}{n-2}, \dots, \frac{n-1}{n-2}, 1\}$ is a subalgebra of L .

Proposition 2.3. [8, 9, 18] In an MV-algebra, the following properties hold:

- (1) $x \oplus x^* = 1$,
- (2) $x \odot x^* = 0$,
- (3) $x \leq y$ if and only if $x \rightarrow y = 1$,
- (4) $x \odot y \leq x \wedge y$,
- (5) $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$,
- (6) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$,
- (7) $x \leq y$ implies $x \odot z \leq y \odot z$,
- (8) $x \vee y = (x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$,
- (9) $x \leq y \rightarrow x$,
- (10) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$,
- (11) $x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z)$,
- (12) $x \odot y \leq z$ if and only if $x \leq y \rightarrow z$,
- (13) $x \ominus y \leq z$ if and only if $x \leq y \oplus z$, for all $x, y, z \in L$.

MV-algebras are non-idempotent generalizations of Boolean algebras. Indeed, Boolean algebras are just the MV-algebras obeying the additional identity $x \oplus x = x$ or $x \odot x = x$. Let L be an MV-algebra and $B(L) = \{a \in L | a \oplus a = a\} = \{a \in L | a \odot a = a\}$ be the set of all idempotent elements of L . Then $(B(L), \oplus, *, 0)$ is a subalgebra of L , which is called the Boolean center of L [8].

Proposition 2.4. [8] Let L be an MV-algebra. Then the following statements are equivalent:

- (1) $x \in B(L)$,
- (2) $x \oplus y = x \vee y$,
- (3) $x \odot y = x \wedge y$, for any $y \in L$.

Proposition 2.5. [20] Let L be an MV-algebra and $e \in B(L)$. Then the following properties hold:

- (1) $e \wedge (x \odot y) = (e \wedge x) \odot (e \wedge y)$,
- (2) $e \vee (x \odot y) = (e \vee x) \odot (e \vee y)$,
- (3) $e \wedge (x \oplus y) = (e \wedge x) \oplus (e \wedge y)$,
- (4) $e \vee (x \oplus y) = (e \vee x) \oplus (e \vee y)$,
- (5) $e \odot (x \rightarrow y) = e \odot [(e \odot x) \rightarrow (e \odot y)]$, for any $x, y \in L$.

Let L be an MV-algebra. A nonempty subset I of L is called an *ideal* of L if it satisfies: (1) $x, y \in I$ implies $x \oplus y \in I$; (2) $x \in I$, $y \in L$ and $y \leq x$ implies $y \in I$. An ideal I of an MV-algebra L is *proper* if $I \neq L$. A proper ideal I of an MV-algebra L is called a prime ideal if for any $x, y \in L$ such that $x \wedge y \in I$, then $x \in I$ or $y \in I$. A nonempty subset I of an MV-algebra L is called a *lattice ideal* of L if it satisfies: (i) for all $x, y \in I$, $x \vee y \in I$; (ii) for all $x, y \in L$, if $x \in I$ and $y \leq x$, then $y \in I$. That is, a lattice ideal of an MV-algebra L is the notion of ideal in the underlying lattice (L, \wedge, \vee) . For any nonempty subset H of L , the smallest lattice ideal containing H is called the lattice ideal generated by H . The lattice ideal generated by H will be denoted by $[H]$. In particular, if $H = \{t\}$, we write $[t]$ for $\{[t]\}$, $[t]$ is called a principal lattice ideal of L . It is easy to check that $[t] = \downarrow t = \{x \in L | x \leq t\}$. From the above and the Principle of Duality, we can define the lattice filter and the principal lattice filter of an MV-algebra L . [10, 11, 14].

Let I be an ideal of an MV-algebra L . We define a binary relation θ_I on L as follows: for any $x, y \in L$, $(x, y) \in \theta_I$ if and only if $(x \ominus y) \oplus (y \ominus x) \in I$. Then, θ_I is a congruence relation on L . Thus, the binary relation \leq on L/I which is defined by $[x] \leq [y]$, if and only if $x \rightarrow y \in I$, is an order relation on L/F . For any $x \in L$, let $[x]_I$ be the equivalence class $[x]_{\theta_I}$ and $L/F = L/\theta_I = \{[x]_I | x \in L\}$. Then L/I becomes an MV-algebra with the natural operations induced from those of L . For more details about ideals in MV-algebras [20].

Definition 2.6. [20] Given ordered sets E, F and order-preserving mappings $f : E \rightarrow F$ and $g : F \rightarrow E$, we say that the pair (f, g) establishes a *Galois connection* between E and F if $fg \geq id_F$ and $gf \leq id_E$.

3. On derivations of MV-algebras

In this section, we investigate some derivations in an MV-algebra. Then, we give some characterizations of additive derivations. Also, we discuss the relationship between additive derivations and ideals of MV-algebras..

Definition 3.1. [2] Let L be an MV-algebra. A map $d : L \rightarrow L$ is called a *derivation* on L if it satisfies the following conditions: for any $x, y \in L$,

$$d(x \odot y) = (d(x) \odot y) \oplus (x \odot d(y)).$$

Now, we present some examples for derivations on MV-algebras.

Example 3.2. Let L be an MV-algebra. Define a map $d : L \rightarrow L$ by $d(x) = 0$ for all $x \in L$, then d is a derivation on L , which is called a zero derivation.

Example 3.3. Let $L = \{0, a, b, c, d, 1\}$ and operations \oplus and $*$ be defined as follows:

\oplus	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	c	d	c	1	1
b	b	d	b	1	d	1
c	c	c	1	c	1	1
d	d	1	d	1	1	1
1	1	1	1	1	1	1

$*$	0	a	b	c	d	1
1	d	c	b	a	0	0

Then $(\{0, a, b, c, d, 1\}, \oplus, *, 0)$ is an MV-algebra. Define a map $d : L \rightarrow L$ by $d(0) = d(a) = d(c) = 0$, $d(b) = d(d) = d(1) = b$. One can check that d is a derivation on L .

Proposition 3.4. [2] Let L be an MV-algebra and d be a derivation on L . Then we have: for any $x, y \in L$,

- (1) $d(0) = 0$,
- (2) $d(1) \in B(L)$,
- (3) $d(x) \odot x^* = x \odot d(x^*) = 0$,
- (4) $d(x) \leq x$,
- (5) $d(x) = d(x) \oplus (x \odot d(1))$.

Definition 3.5. [2] Let L be an MV-algebra and d be a derivation on L .

- (1) d is called an *isotone derivation* provided that $x \leq y$ implies $d(x) \leq d(y)$ for all $x, y \in L$,
- (2) d is called an *additive derivation* provided that $d(x \oplus y) = d(x) \oplus d(y)$ for all $x, y \in L$.

Example 3.6. Considering the MV-algebra $S_4 = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$ in Example 2.2. Define a map $d : S_4 \rightarrow S_4$ by $d(0) = d(1) = d(\frac{1}{3}) = 0$, $d(\frac{2}{3}) = \frac{1}{3}$. One can check that d is a derivation on S_4 , but it is not an additive derivation on L , since $d(\frac{1}{3} + \frac{2}{3}) = d(1) = 0 \neq \frac{1}{3} = d(\frac{1}{3}) + d(\frac{2}{3})$. Also, d is not an isotone derivation on L , since $\frac{2}{3} \leq 1$, but $d(\frac{2}{3}) = \frac{1}{3} \geq 0 = d(1)$.

Example 3.7. Considering the derivation d in Example 3.3, one can see that d is not only an additive and but also is an isotone derivation on L .

Proposition 3.8. Let L be an MV-algebra and d be an additive derivation on L . Then we have: for any $x, y \in L$,

- (1) d is an isotone derivation,

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 6 (2) $d(x) = d(1) \odot x$,
 7 (3) $dd(x) = d(x)$,
 8 (4) $d(x) \in B(L)$,
 9 (5) $d(d(x) \rightarrow d(y)) = d(x \rightarrow y)$,
 10 (6) $Fix_d(L) = d(L)$, where $Fix_d(L) = \{x \in L | d(x) = x\}$,
 11 (7) if $d(L) = L$, then $d = id_L$,
 12 (8) $\text{Ker}(d)$ is an ideal of L , where $\text{Ker}(d) = \{x \in L | d(x) = 0\}$.
 13
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16 **Proof.** (1) If $x \leq y$, then $y = x \vee y = x \oplus (x^* \odot y)$. From Definition 3.5(2), we
 17 have $d(y) = d(x \oplus (x^* \odot y)) = d(x) \oplus d(x^* \odot y) \geq d(x)$. Thus, $d(x) \leq d(y)$.
 18
 19 (2) From Proposition 3.4(2),(4) and (1), we have $d(x) = d(1) \wedge d(x) = d(1) \odot$
 20 $d(x) \leq d(1) \odot x$. On the other hand, we conclude from Definition 3.1 that $d(x) \geq$
 21 $x \odot d(1)$. Thus, $d(x) = d(1) \odot x$.
 22
 23 (3) From Proposition 3.4(2) and (2), we have $dd(x) = d(d(1) \odot x) = d(1) \odot$
 24 $d(1) \odot x = d(1) \odot x = d(x)$.
 25
 26 (4) From Definition 3.1 and (2), we have $d(x) = d(x \odot 1) = (d(x) \odot 1) \oplus (x \odot$
 27 $d(1)) = d(x) \oplus (x \odot d(1)) = d(x) \oplus d(x)$. Thus, $d(x) \in B(L)$.
 28
 29 (5) From Proposition 2.5(5), 3.4(2) and (2), we have $d(d(x) \rightarrow d(y)) = d(1) \odot$
 30 $[(d(1) \odot x) \rightarrow (d(1) \odot y)] = d(1) \odot (x \rightarrow y) = d(x \rightarrow y)$. Thus, $d(d(x) \rightarrow d(y)) =$
 31 $d(x \rightarrow y)$.
 32
 33 (6) Let $y \in d(L)$. So there exists $x \in L$ such that $y = d(x)$ and hence $d(y) =$
 34 $dd(x) = d(x) = y$, that is, $y \in Fix_d(L)$. Conversely, if $y \in Fix_d(L)$, then we have
 35 $y \in d(L)$. Therefore, $Fix_d(L) = d(L)$.
 36
 37 (7) For any $x \in L$, we have $x = d(x_0)$ for some $x_0 \in L$. From (3), we have
 38 $d(x) = dd(x_0) = d(x_0) = x$. Therefore, $d = id_L$.
 39
 40 (8) From Proposition 3.4(1), we have $d(0) = 0$, that is, $0 \in \text{Ker}(d)$. If $x, y \in$
 41 $\text{Ker}(d)$, then $d(x) = d(y) = 0$ and hence $d(x \oplus y) = d(x) \oplus d(y) = 0$, that is, $x \oplus y \in$
 42 $\text{Ker}(d)$. Finally, if $x \leq y$ and $y \in \text{Ker}(d)$, then $d(x) \leq d(y) = 0$ and hence $d(x) = 0$,
 43 that is, $x \in \text{Ker}(d)$.

44 **Remark 3.9.** In [2], the results (1) and (8) in Proposition 3.8 are proved as
 45 Theorems 3.16, 3.17 under the additional condition that L is an linearly ordered
 46 MV-algebra, but this condition is redundant as our proof shows.
 47
 48

49 **Theorem 3.10.** Let L be an MV-algebra and d be a derivation on L . Then the
 50 following statements are equivalent:
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 52

53 (1) d is an additive derivation,
 54 (2) d is an isotone derivation,
 55 (3) $d(x) \leq d(1)$,
 56 (4) $d(x) = d(1) \odot x$,
 57 (5) $d(x \odot y) = d(x) \odot y = x \odot d(y)$,
 58 (6) $d(x \wedge y) = d(x) \wedge d(y)$,
 59 (7) $d(x \vee y) = d(x) \vee d(y)$,
 60 (8) $d(x \odot y) = d(x) \odot d(y)$,
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 62
 63
 64
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(9) $d(x) \leq y$ if and only if $d(x) \leq d(y)$,

(10) $d(x) \rightarrow d(y) = d(x) \rightarrow y$.

Proof. (1) \Rightarrow (2) It follows from Proposition 3.8(1)

(2) \Rightarrow (3) It is straightforward.

(3) \Rightarrow (4) It follows from Proposition 3.8(2).

(4) \Rightarrow (1) From Proposition 2.5(4) and Proposition 3.8(2), we have $d(x \oplus y) = d(1) \odot (x \oplus y) = (d(1) \odot x) \oplus (d(1) \odot y) = d(x) \oplus d(y)$, that is, $d(x \oplus y) = d(x) \oplus d(y)$.

From the above proof, one can see the statements (1)-(4) are equivalent.

(4) \Rightarrow (5) Let $d(x) = d(1) \odot x$ for all $x \in L$. It follows that $d(x \odot y) = d(1) \odot x \odot y = x \odot (d(1) \odot y)$, that is, $d(x \odot y) = d(x) \odot y = x \odot d(y)$ for any $x, y \in L$.

(5) \Rightarrow (4) Taking $y = 1$ in (5), we have $d(x) = d(1) \odot x$ for any $x \in L$.

(4) \Rightarrow (6) Let $d(x) = d(1) \odot x$ for any $x \in L$. It follows that $d(x \wedge y) = d(1) \odot (x \wedge y) = d(1) \wedge (x \wedge y) = d(1) \wedge d(1) \wedge x \wedge y = d(x) \wedge d(y)$ for any $x, y \in L$.

(4) \Rightarrow (7) Since $(L, \wedge, \vee, 0, 1)$ is a distributive lattice. From (4), we have $d(x \vee y) = d(1) \odot (x \vee y) = d(1) \wedge (x \vee y) = (d(1) \wedge x) \vee (d(1) \wedge y) = (d(1) \odot x) \vee (d(1) \odot y) = d(x) \vee d(y)$ for any $x, y \in L$.

(7) \Rightarrow (2) Let $x \leq y$, we have $x \vee y = y$. From (7), we have $d(y) = d(x \vee y) = d(x) \vee d(y) \geq d(x)$ for any $x, y \in L$. Therefore, d is an isotone derivation.

(4) \Rightarrow (8) From (4), we have $d(x \odot y) = d(1) \odot (x \odot y) = d(1) \odot d(1) \odot x \odot y = d(x) \odot d(y)$ for any $x, y \in L$.

(6) \Rightarrow (3), (7) \Rightarrow (3), (8) \Rightarrow (3) are straightforward.

(2) \Rightarrow (9) For all $x, y \in L$, assume that $d(x) \leq y$, we have $dd(x) \leq d(y)$. By 3.8(2), we get $dd(x) = d(x)$. Thus $d(x) \leq d(y)$. Conversely, suppose that $d(x) \leq d(y)$, we get $d(x) \leq d(y) \leq y$ for all $x, y \in L$.

(9) \Rightarrow (2) Let $x \leq y$, we have $d(x) \leq x \leq y$ and hence $d(x) \leq d(y)$ for all $x, y \in L$.

(2) \Rightarrow (10) Suppose that d is an isotone derivation on L . From $d(y) \leq y$ for any $y \in L$, it follows that $d(x) \rightarrow d(y) \leq d(x) \rightarrow y$. On the other hand, let $t \leq d(x) \rightarrow y$ for all $t \in L$, we can obtain $d(x) \odot t \leq y$. Since d is an isotone derivation, we have $d(d(x) \odot t) \leq d(y)$ for all $x, y, t \in L$. From $d(x \odot y) = (d(x) \odot y) \oplus (x \odot d(y))$, we get $d(x) \odot y \leq d(x \odot y)$ for all $x, y \in L$. It follows $d(d(x)) \odot t \leq d(d(x) \odot t)$. By Proposition 3.8(3), we have $d(x) \odot t \leq d(d(x) \odot t) \leq d(y)$. Hence $t \leq d(x) \rightarrow d(y)$ for all $t \in L$, which implies $d(x) \rightarrow y \leq d(x) \rightarrow d(y)$ for all $x, y \in L$. Therefore, we obtain $d(x) \rightarrow d(y) = d(x) \rightarrow y$ for any $x, y \in L$.

(10) \Rightarrow (2) Assume that $d(x) \rightarrow d(y) = d(x) \rightarrow y$ for all $x, y \in L$. For any $x, y \in L$, let $x \leq y$, by Proposition 3.4(4), we have $d(x) \odot 1 = d(x) \leq x \leq y$. It follows that $1 \leq d(x) \rightarrow y = d(x) \rightarrow d(y)$, which implies $d(x) \leq d(y)$.

Remark 3.11. (1) From the above theorem, one can see that isotone derivations are equivalent to additive derivations on MV-algebras.

(2) From the Example 3.3, one can check that an additive derivation is not a homomorphism on an MV-algebra L in general, since $d(a^*) = b \neq 1 = (d(a))^*$.

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6 (3) From the Example 3.3, one can check that the fixed point set of an additive
7 derivation d is not a subalgebra of an MV-algebra L in general, since $0^* = 1 \notin$
8 $\{0, b\} = Fix_d(L)$.
9
10 (4) From (4) of the above theorem, one can see that every additive derivation d
11 on an MV-algebra L is completely defined by the image $d(1)$ of the 1.
12
13 (5) It is easily seen that every additive derivation is a lattice derivation in the
14 sense of Ferrari [12], which is a unary map $d : L \rightarrow L$ satisfying conditions
15 Theorem 3.10(7) and $d(x \wedge y) = (d(x) \wedge y) \vee (x \wedge d(y))$. However, the converse
16 is not true in general. Moreover, Ghorbain proved that a lattice derivation on
17 an MV-algebra is an additive derivation if and only if $d(L) \subseteq B(L)$ in [13].
18
19

20 The following theorem shows that the fixed point set $Fix_d(L)$ of additive derivations
21 in an MV-algebra L has the same structure as L , which reveals the essence
22 of the fixed point set of additive derivations.
23
24

25 **Theorem 3.12.** Let L be an MV-algebra and d be an additive derivation on L .
26 Then $(Fix_d(L), \oplus, \neg, 0)$ is an MV-algebra, where $\neg x = d(x^*) = d(1) \odot x^* = (x \oplus$
27 $(d(1))^*)^*$ for any $x \in Fix_d(L)$.
28

29 **Proof.** First, we will prove that $(Fix_d(L), \oplus, 0)$ is a commutative monoid. From
30 Definition 3.5(2), we have $Fix_d(L)$ is closed under \oplus . It follows that $(Fix_d(L), \oplus)$
31 is commutative semigroup. For all $x \in Fix_d(L)$, we can obtain that $x \oplus 0 = x$,
32 that is, 0 is a unital element.
33
34

35 Next, we will prove that $Fix_d(L)$ is closed under \neg . For all $x \in Fix_d(L)$, from
36 Proposition 3.4(2) and 3.8(2), we have $d(\neg x) = d(d(x^*)) = d(1) \odot d(1) \odot x^* =$
37 $d(1) \odot x^* = d(x^*) = \neg x$, that is, $Fix_d(L)$ is closed under \neg .
38

39 Finally, we will verify the remaining axioms of an MV-algebra.

40 (MV2) For all $x \in Fix_d(L)$, from Theorem 3.10(3), we have $\neg\neg x = (\neg x \oplus$
41 $(d(1))^*)^* = ((x \oplus (d(1))^*) \oplus (d(1))^*)^* = (x \oplus (d(1))^*) \odot d(1) = x \wedge d(1) = d(x) \wedge d(1) =$
42 $d(x) = x$, that is, $\neg\neg x = x$.
43

44 (MV3) For all $x \in Fix_d(L)$, from Theorem 3.10(3), we have $x \oplus \neg 0 = x \oplus d(1) =$
45 $d(x) \oplus d(1) = d(x) \vee d(1) = d(1)$, that is, $x \oplus \neg 0 = d(1)$.
46

47 (MV4) For any $x, y \in Fix_d(L)$, we have $\neg(\neg x \oplus y) \oplus y = (\neg x \oplus y \oplus (d(1))^*)^* \oplus y =$
48 $(x \oplus (d(1))^*)^* \oplus y \oplus (d(1))^* \oplus y = ((x \oplus (d(1))^*) \odot d(1) \odot y^*) \oplus y = ((x \wedge d(1)) \odot$
49 $y^*) \oplus y = x \vee y$. in the similar way, one can prove that $\neg(x \oplus \neg y) \oplus x = x \vee y$, and
50 hence $\neg(\neg x \oplus y) \oplus y = \neg(x \oplus \neg y) \oplus x$.
51

52 Therefore, we obtain that $(Fix_d(L), \oplus, \neg, 0)$ is an MV-algebra.
53

54 **Theorem 3.13.** Let L be an MV-algebra and d an additive derivation on L . Then
55 we have the following properties:
56

57 (1) $d : L \longrightarrow Fix_d(L)$ is a surjective homomorphism,
58 (2) $\bar{d} : L/Ker(d) \longrightarrow Fix_d(L)$ is an isomorphism.
59

60 **Proof.** (1) It follows from Theorem 3.10 and 3.12.
61

62 (2) From Proposition 3.8(8), we obtain that $Ker(d)$ is an ideal of L . Let
63 $x \sim_{Ker(d)} y$. Then $(x \ominus y) \oplus (y \ominus x) \in Ker(d)$, which implies $d((x \ominus y) \oplus (y \ominus$
64 $x)) = 0$. Hence $d(x) = d(y)$.
65

6 $x)) = d(x \ominus y) \oplus d(y \ominus x) = 0$. Furthermore, from Theorem 3.10(8), we have
 7 $d(x \ominus y) = d(x) \odot d(y^*) = d(x) \odot y^* = d(y \ominus x) = d(y) \odot x^* = 0$, that is, $d(x) \leq y$
 8 and $x \leq d(y)$. Further by Theorem 3.10(9), we have $d(x) = d(y)$. Thus, \bar{d} is
 9 well defined. Moreovwe, it follows from (1) that $\bar{d} : L/Ker(d) \longrightarrow Fix_d(L)$ is an
 10 isomorphism.
 11

12 **Theorem 3.14.** Let L be an MV-algebra and $d : L \rightarrow L$ be a map on L such that
 13 $d(L) \subseteq B(L)$. Then the following statements are equivalent:
 14

- 15 (1) d is an additive derivation on L ,
- 16 (2) $d(x) = d(1) \odot x$,
- 17 (3) $d(x \odot y) = d(x) \odot y = x \odot d(y)$.

18 **Proof.** (1) \Rightarrow (2) It follows from Proposition 3.8(2).
 19

20 (2) \Rightarrow (3) The proof of that is similar to Theorem 3.10 (4) \Rightarrow (5).
 21

22 (3) \Rightarrow (1) Taking $y = 1$ in (3), we have $d(x) = d(1) \odot x$. From $d(L) \subseteq B(L)$,
 23 we have $d(x \odot y) = d(x \odot y) \oplus d(x \odot y) = (d(x) \odot y) \oplus (x \odot d(y))$, that is, d is a
 24 derivation on L . Moreover, from Theorem 3.10 (1) \Leftrightarrow (4), we obtain that d is an
 25 additive derivation on L .
 26

27 **Corollary 3.15.** Let L be a Boolean algebra and $d : L \rightarrow L$ be a map on L . Then
 28 the following statements are equivalent:
 29

- 30 (1) d is an additive derivation on L ,
- 31 (2) $d(x) = d(1) \wedge x$,
- 32 (3) $d(x \wedge y) = d(x) \wedge y = x \wedge d(y)$.

33 **Proof.** It follows from Theorem 3.14.
 34

35 **Theorem 3.16.** Let L be an MV-algebra and d be an additive derivation on L .
 36 Then the following statements are equivalent:
 37

- 38 (1) d is an identity map,
- 39 (2) $d(x^*) = (d(x))^*$,
- 40 (3) $d(1) = 1$,
- 41 (4) d is homorphism,
- 42 (5) d is surjection,
- 43 (6) d is one to one,
- 44 (7) d is isomorphic.

45 **Proof.** (1) \Rightarrow (2) It is straightforward.
 46

47 (2) \Rightarrow (3) From Proposition 3.4(1), we have $d(1) = d(0^*) = (d(0))^* = 0^* = 1$.
 48

49 (3) \Rightarrow (1) From Proposition 3.8(2), we have $d(x) = d(1) \odot x = 1 \odot x = x$.
 50

51 (2) \Rightarrow (4) From Proposition 3.4(1) and Theorem 3.10(8), we obtain that d is
 52 homomorphism.
 53

54 (4) \Rightarrow (2) It is straightforward.
 55

56 (1) \Rightarrow (5) It is straightforward.
 57

(5) \Rightarrow (1) Suppose that d is surjection, then there exists $x \in L$ such that $d(x) = 1$. From Proposition 3.4(4), we have $1 = d(x) \leq x$ and hence $x = 1$, that is, $d(1) = 1$. Further from (3) \Rightarrow (1), we get that d is an identity map.

(1) \Rightarrow (6) It is straightforward.

(6) \Rightarrow (1) Suppose that d is one to one, by Proposition 3.8(2), we have $d((d(1))^*) = d(1) \odot (d(1))^* = 0$ and hence $d(1) = 1$. hence $x = 1$, that is, $d(1) = 1$. From (3) \Rightarrow (1), we get that d is an identity map. From (3) \Rightarrow (1), we get that d is an identity map.

(1) \Leftrightarrow (7) It follows from (1) \Leftrightarrow (4), (1) \Leftrightarrow (5), (1) \Leftrightarrow (6).

In what follows, we discuss the relationship between ideals and derivations on MV-algebras.

The following example shows that the fixed point set $\text{Fix}_d(L)$ of a derivation d is not an ideal of an MV-algebra L .

Example 3.17. Considering $S_3 = \{0, \frac{1}{2}, 1\}$ in Example 2.2 and defining a map $d : S_3 \longrightarrow S_3$ by $d(0) = d(1) = 0$, $d(\frac{1}{2}) = \frac{1}{2}$. One can check that d is a derivation, while it is not an additive derivation, and $\text{Fix}_d(L) = \{0, \frac{1}{2}\}$ is not an ideal of L since $\frac{1}{2} \oplus \frac{1}{2} = 1 \notin \text{Fix}_d(L)$.

However, if the conditions are strengthened, we can obtain the following result.

Proposition 3.18. Let $(L, \oplus, *, 0, 1)$ be an MV-algebra and d be an additive derivation on L . Then $\text{Fix}_d(L)$ is an ideal of L .

Proof. It follows from Theorem 3.10 (1) \Leftrightarrow (2) and Proposition 3.23 (3) in [13].

The following is a counterexample showing that the converse of Proposition 3.18 may not hold.

Example 3.19. Considering the Example 3.6, one can check that $\text{Fix}_d(L) = \{0\}$ is an ideal of L . However, d is not an additive derivation on L .

Inspired by Proposition 3.18, it is natural to ask that whether there exists an additive derivation d such that $\text{Fix}_d(L) = I$ for given ideal I in an MV-algebra L .

According to Theorem 4.13 in [21], we give the positive answer under certain conditions to the above question.

Proposition 3.20. Let L be a Boolean algebra and I be a non-void prime ideal of L . Then there exists an additive derivation d such that $\text{Fix}_d(L) = I$. Indeed, a map $d : L \longrightarrow L$ that is defined by: for all $t, x \in L$,

$$d(x) = \begin{cases} x, & x \in I \\ x \wedge t, & x \in L/I, \end{cases}$$

is an additive derivation satisfying $\text{Fix}_d(L) = I$.

In fact, the above proposition shows under some suitable conditions, there exists an additive derivation on an MV-algebra L such that $\text{Fix}_d(L) = I$ for given ideal I of L . Furthermore, we have the following **open problem**:

- (1) For any ideal I of an MV-algebra L , whether there exists an additive derivation d such that $\text{Fix}_d(L) = I$.

4. Boolean derivation and their applications

In this sections, we investigate Boolean additive derivations and their adjoint derivations. In particular, we prove that every MV-algebra is isomorphic to the direct product of the fixed point set of Boolean additive derivations and that of their adjoint derivations. Finally, we show that the structure of a Boolean algebra is completely determined by its set of all Boolean additive (implicative) derivations.

In what follows, let L be an MV-algebra and $a \in L$, we define four maps as follows:

- (1) $d_a : L \rightarrow L$ such that $d_a(x) = x \odot a$, for all $x \in L$,
- (2) $d_{a^*} : L \rightarrow L$ such that $d_{a^*}(x) = x \odot a^* = x \ominus a$, for all $x \in L$,
- (3) $g_a : L \rightarrow L$ such that $g_a(x) = a \oplus x$, for all $x \in L$,
- (4) $g_{a^*} : L \rightarrow L$ such that $g_{a^*}(x) = a^* \oplus x = a \rightarrow x$, for all $x \in L$.

Theorem 4.1. Let L be an MV-algebra. Then the following statements are equivalent:

- (1) L is a Boolean algebra,
- (2) d_a is an additive derivation on L , for all $a \in L$,
- (3) d_{a^*} is an additive derivation on L , for all $a \in L$.

Proof. (1) \Rightarrow (2) If L is a Boolean algebra, then we get that $x \oplus y = x \vee y$ for any $x, y \in L$. Further by Corollary 3.15, we have d_a is an additive derivation on L .

(2) \Rightarrow (1) Assume that d_a is an additive derivation on L . From Definition 3.1, we have $a = d_a(1) = d_a(1 \odot 1) = (d_a(1) \odot 1) \oplus (1 \odot d_a(1)) = a \oplus a$ for any $a \in L$ and hence $L \subseteq B(L)$. Therefore, L is a Boolean algebra.

(1) \Leftrightarrow (3) The proof of (1) \Leftrightarrow (3) is similar to that of (1) \Leftrightarrow (2).

Remark 4.2. From Theorem 4.1, we obtain that d_a is an additive derivation on an MV-algebra if and only if $a \in B(L)$. Based on the above consideration, we called d_a a Boolean additive derivation on an MV-algebra L .

Next, we will discuss the adjoint derivation of Boolean additive derivations. First, we introduce new derivation on MV-algebras.

Definition 4.3. Let L be an MV-algebra. A map $g : L \rightarrow L$ is called an *implication derivation* on L if it preserves \rightarrow and satisfies the following conditions: for any $x, y \in L$,

$$g(x \rightarrow y) = (g(x) \rightarrow y) \oplus (x \rightarrow g(y)).$$

Example 4.4. Let $L = \{0, a, b, 1\}$ be a chain and operations \oplus and $*$ be defined as follows:

\oplus	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

$*$	0	a	b	1
1	b	a	0	0

Then $(\{0, a, b, 1\}, \oplus, *, 0)$ is an MV-algebra. Define a map $g : L \rightarrow L$ by $g(0) = g(a) = a$, $g(b) = g(1) = 1$. One can check that g is an implicative derivation on L , while g is not a homomorphism on L , since $g(0) \neq 0$.

Theorem 4.5. Let L be an MV-algebra. Then the following statements are equivalent:

- (1) L is a Boolean algebra,
- (2) g_{a^*} is an implicative derivation on L , for all $a \in L$,
- (3) g_a is an implicative derivation on L , for all $a \in L$.

Proof. (1) \Rightarrow (2) Let L be a Boolean algebra and $a \in L$. First, we will prove that $g_{a^*}(x \rightarrow y) = g_{a^*}(x) \rightarrow g_{a^*}(y)$. In particular, since $(a \odot x \rightarrow y) \odot a \odot (a \rightarrow x) \leq (a \odot x \rightarrow y) \odot a \odot x \leq y$, we have $a \odot x \rightarrow y \leq a \odot (a \rightarrow x) \rightarrow y$ and hence $g_{a^*}(x \rightarrow y) = a \rightarrow (x \rightarrow y) = x \rightarrow (a \rightarrow y) = a \odot x \rightarrow y \leq a \odot (a \rightarrow x) \rightarrow y = (a \rightarrow x) \rightarrow (a \rightarrow y) = g_{a^*}(x) \rightarrow g_{a^*}(y)$. On the other hand, since $x \leq a \rightarrow x$, it is easy to show that $a \odot x \leq a \odot (a \rightarrow x)$ and hence that $a \odot (a \rightarrow x) \rightarrow y \leq a \odot x \rightarrow y$, that is, $g_{a^*}(x) \rightarrow g_{a^*}(y) \leq g_{a^*}(x \rightarrow y)$. Thus, we have $g_{a^*}(x \rightarrow y) = g_{a^*}(x) \rightarrow g_{a^*}(y)$.

Next, from Proposition 2.5, we have $g_{a^*}(x \rightarrow y) = a \oplus (x \rightarrow y) = x^* \oplus g_{a^*}(y) = ((g_{a^*}(x)) \vee x^*) \oplus (g_{a^*}(y) \vee y) = ((g_{a^*}(x))^* \oplus y) \oplus (x^* \oplus g_{a^*}(y)) = (g_{a^*}(x) \rightarrow y) \oplus (x \rightarrow g_{a^*}(y))$.

Combining them, one can see that g_{a^*} is an implicative derivation on L .

(2) \Rightarrow (1) Assume that g_{a^*} is an implicative derivation on L , for any $a \in L$. Then $g_{a^*}(x \rightarrow y) = g_{a^*}(x) \rightarrow g_{a^*}(y)$ holds for all $x, y \in L$, then taking $x = a$ and $y = a \odot a$ we have $a \rightarrow (a \rightarrow a \odot a) = (a \rightarrow a) \rightarrow (a \rightarrow a \odot a)$. Since $a \rightarrow (a \rightarrow a \odot a) = a \odot a \rightarrow a \odot a = 1$, we have $1 = 1 \rightarrow (a \rightarrow a \odot a) = a \rightarrow a \odot a$. This implies that $a \leq a \odot a$ and hence that $a = a \odot a$, that is L is a Boolean algebra.

(1) \Leftrightarrow (3) The proof of (1) \Leftrightarrow (3) is similar to that of (1) \Leftrightarrow (2).

Remark 4.6. From Theorem 4.5, we obtain that g_a is an implication derivation on an MV-algebra if and only if $a \in B(L)$. Based on the above consideration, we called g_a a Boolean implication derivation on an MV-algebra L .

Definition 4.7. Let L be an MV-algebra and μ_a be a Boolean additive derivation on L . The Boolean additive derivation μ_a is called residuated if there exists a Boolean implicative derivation ν_a on L such that a pair (μ_a, ν_a) forms a Galois connection.

The Boolean implicative derivation ν_a is called the adjoint derivation of the Boolean additive derivation μ_a .

From the notion of Galois connections, we have that if the Boolean additive derivation μ_a is residuated, then μ_a and its adjoint derivation must be isotone. In particular, if the Boolean additive derivation μ_a has the adjoint derivation ν_a , then the adjoint of μ_a unique. Therefore, we shall denote this unique ν_a by μ_a^* .

Example 4.8. Consider the MV-algebra in Example 4.7. Define two maps $\nu_b(0) = \nu_b(b) = b$, $\nu_b(a) = \nu_b(1) = 1$, $\mu_b(0) = \mu_b(a) = 0$, $\mu_b(b) = \mu_b(1) = b$, respectively. One can check that ν_b and μ_b are Boolean implicative derivation and Boolean additive derivation on L , respectively. Further, one can check that (ν_b, μ_b) forms a Galois connection.

Theorem 4.9. Let L be an MV-algebra and $a \in B(L)$. The map $g_a(g_a^*)$ is the adjoint derivation of the Boolean additive derivation $d_a^*(d_{a^*}^*)$ on L , that is, $d_a^* = g_{a^*}(d_{a^*}^* = g_a)$.

Proof. First, from Theorems 4.1 and 4.5, we get d_a and g_{a^*} are Boolean additive derivation and Boolean implicative derivation on L , respectively. Moreover, for all $x, y \in L$, let $x \leq y$, we conclude from Proposition 2.3 that $g_{a^*}(x) = a \rightarrow x \leq a \rightarrow y = g_{a^*}(y)$, that is, g_{a^*} is isotone. For all $x, y \in L$, we have $d_a(x) = a \odot x \leq y$ if and only if $x \leq a \rightarrow y = g_{a^*}(y)$. Thus, (d_a, g_{a^*}) forms a Galois connection. Combining them, we obtain that g_{a^*} is the adjoint derivation of d_a , that is, $d_a^* = g_{a^*}$. In the similar way, one can prove that $d_{a^*}^* = g_a$.

In what follows, we denote by $Fix_{d_a}(L)$ the set of all fixed points for d_a and $Fix_{g_a}(L)$ the set of all fixed points of L for g_a , respectively.

Theorem 4.10. Let L be an MV-algebra and $a \in B(L)$. Then $(Fix_{d_a}(L), \oplus, \neg_1, 0, a)$ is an MV-algebra, where $\neg_1 x = d_a(x^*)$, for all $x \in L$.

Proof. The proof is similar to that of Theorem 3.12.

Theorem 4.11. Let L be an MV-algebra and $a \in B(L)$. Then $(Fix_{g_a}(L), \oplus, \circ_1, a, 1)$ is an MV-algebra, where $x^{\circ_1} = g_a(x^*)$, for all $x \in L$.

Proof. First, we prove that $(Fix_{g_a}(L), \oplus, a)$ is a commutative monoid. One can easily to check that $Fix_{g_a}(L)$ is closed under \oplus . It follows that $(Fix_{g_a}(L), \oplus)$ is commutative semigroup. For all $x \in Fix_{g_a}(L)$, we can obtain that $x \oplus a = x$, that is, a is a unital element.

6 Next, we will prove that $(Fix_{g_a}(L))$ is closed under \circ_1 . For all $x \in Fix_{g_a}(L)$, we
7 have $g(x^{\circ_1}) = g(g(x^*)) = a \oplus a \oplus x^* = a \oplus x^* = g(x^*) = x^{\circ_1}$, that is, $Fix_{g_a}(L)$ is
8 closed under \circ_1 .

9 Finally, we will verify the remaining axioms of an MV-algebra.

10 (MV2) For all $x \in Fix_{g_a}(L)$, we have $x^{\circ_1 \circ_1} = (x^{\circ_1} \odot (a^*))^* = ((x \odot a)^*)^* \oplus a^* =$
11 $(x \odot a^*) \oplus a = x \vee a = g_a(x) \vee g_a(a) = d(x) = x$, that is, $x^{\circ_1 \circ_1} = x$.

12 (MV3) For all $x \in Fix_{g_a}(L)$, we have $x \oplus a^{\circ_1} = x \oplus 1 = 1$.

13 (MV4) For any $x, y \in Fix_{g_a}(L)$, we have $(x^{\circ_1} \oplus y)^{\circ_1} \oplus y = (x^{\circ_1} \odot y \odot a^*)^* \odot y =$
14 $(x \odot a^*)^* \odot y \odot a^* = ((x \odot a^*) \oplus a \oplus y^*) \odot y = ((x \vee a) \oplus y^*) \odot y = x \wedge y$. In the
15 similar way, one can prove that $(x^{\circ_1} \odot y^{\circ_1})^{\circ_1} \odot x = x \wedge y$, and hence $(x^{\circ_1} \oplus y)^{\circ_1} \oplus y =$
16 $(x \oplus y^{\circ_1})^{\circ_1} \oplus x$.

17 Therefore, we obtain that $(Fix_{g_a}(L), \oplus, \circ_1, a, 1)$ is an MV-algebra.

18 **Corollary 4.12.** Let L be an MV-algebra and $a \in B(L)$. Then $(Fix_{d_a}(L), \oplus, \neg_2, 0, a^*)$
19 is an MV-algebra, where $\neg_2 x = d_{a^*}(x^*)$, for all $x \in L$.

20 **Proof.** The proof is similar to that of Theorem 4.10.

21 **Corollary 4.13.** Let L be an MV-algebra and $a \in B(L)$. Then $(Fix_{g_{a^*}}(L), \oplus, \circ_2, a^*, 1)$
22 is an MV-algebra, where $x^{\circ_2} = g_{a^*}(x^*)$, for all $x \in L$.

23 **Proof.** The proof is similar to that of Theorem 4.11.

24 The following theorem shows the relationship between $Fix_{d_a}(L)$ and $Fix_{g_{a^*}}(L)$.

25 **Theorem 4.14.** Let L be an MV-algebra and $a \in B(L)$. Then MV-algebras
26 $(Fix_{d_a}(L), \oplus, \neg_1, 0, a)$ and $(Fix_{g_{a^*}}(L), \oplus, \circ_2, a^*, 1)$ are isomorphic.

27 **Proof.** For all $a \in L$, let $f : Fix_{d_a}(L) \longrightarrow Fix_{g_{a^*}}(L)$ be defined by $f(x) = a^* \oplus x$
28 for all $x \in Fix_{d_a}(L)$. Clearly, f is a map from $Fix_{d_a}(L)$ to $Fix_{g_{a^*}}(L)$, that is, f is
29 well defined.

30 (1) For all $x, y \in Fix_{d_a}(L)$, we have $f(x \oplus y) = a^* \oplus (x \oplus y) = a^* \oplus a^* \oplus x \oplus y =$
31 $(a^* \oplus x) \oplus (a^* \oplus y) = f(x) \oplus f(y)$, that is, $f(x \oplus y) = f(x) \oplus f(y)$.

32 (2) For all $x \in Fix_{d_a}(L)$, we have $f(\neg_1 x) = a^* \oplus (a \odot x^*) = a^* \vee (a \wedge x^*) =$
33 $(a^* \vee a) \wedge (a^* \vee x) = a^* \oplus x$. One the other hand, we have $(f(x))^{\circ_2} = (a^* \oplus x)^{\circ_2} =$
34 $a^* \oplus (a^* \oplus x) = a^* \oplus x$ and hence $f(\neg_1 x) = (f(x))^{\circ_2}$.

35 (3) For all $x, y \in Fix_{d_a}(L)$, that is, $x = a \odot x$ and $y = a \odot y$, if $f(x) = f(y)$,
36 then $a^* \oplus x = a^* \oplus y$. From $a \odot x \leq x$, we have $x \leq a \rightarrow x = a^* \oplus x$, then
37 $x \leq a^* \oplus y = a \rightarrow y$. It follows that $a \odot x \leq y$, which implies $x \leq y$. Similarly, we
38 can prove $y \leq x$ and hence $x = y$. Consequently, we obtain that f is injective.

39 (4) One can easily prove the fact that $x^* \vee y = x^* \oplus (x \odot y) = y$ if and only if there
40 exists $z \in L$ such that $x^* \oplus z = y$ for all $x, y \in L$. From the above fact, we will prove
41 that f is surjective. Now, for all $x \in Fix_{g_{a^*}}(L)$, then $x = g_{a^*}(x) = a^* \oplus x$. Using the
42 above fact, we have that $f(a \odot x) = a^* \oplus (a \odot x) = a^* \oplus (a \odot (a^* \oplus x)) = a^* \oplus x = x$.
43 Thus, we conclude that f is surjective. Also, it is easy to verify that $f^{-1}(x) = a \odot x$
44 is a homomorphism from $Fix_{g_{a^*}}(L)$ to $Fix_{d_a}(L)$.

45 Combining them, we obtain that f is an isomorphism from $(Fix_{d_a}(L), \oplus, \neg_1, 0, a)$
46 to $(Fix_{g_{a^*}}(L), \oplus, \circ_2, a^*, 1)$. Therefore, MV-algebras $(Fix_{d_a}(L), \oplus, \neg_1, 0, a)$ and $(Fix_{g_{a^*}}$
47 $(L), \oplus, \circ_2, a^*, 1)$ are isomorphic.

Theorem 4.15. Let L be an MV-algebra and $a \in B(L)$. Then MV-algebras $(Fix_{d_a^*}(L), \oplus, \neg_2, 0, a^*)$ and $(Fix_{g_a}(L), \oplus, \circ_1, a, 1)$ are isomorphic.

Proof. For all $a \in L$, let $u : Fix_{d_a^*}(L) \longrightarrow Fix_{g_a}(L)$ be defined by $u(x) = a \oplus x$ for all $x \in Fix_{d_a^*}(L)$. Clearly, u is a map from $Fix_{d_a^*}(L)$ to $Fix_{g_a}(L)$, that is, u is well defined.

(1) For all $x, y \in Fix_{d_a^*}(L)$, we have $u(x \oplus y) = a \oplus (x \oplus y) = a \oplus a \oplus x \oplus y = (a \oplus x) \oplus (a \oplus y) = u(x) \oplus u(y)$, that is, $u(x \oplus y) = u(x) \oplus u(y)$.

(2) For all $x \in Fix_{d_a^*}(L)$, we have $u(\neg_2 x) = a \oplus (a \odot x^*) = a \vee (a \wedge x^*) = (a \vee a) \wedge (a \vee x) = a \oplus x$. One the other hand, we have $(u(x))^{\circ_1} = (a \oplus x)^{\circ_1} = a \oplus (a \oplus x) = a \oplus x$ and hence $u(\neg_2 x) = (u(x))^{\circ_1}$.

(3) For all $x, y \in Fix_{d_a^*}(L)$, that is, $x = a \ominus x$ and $y = a \ominus y$, if $u(x) = u(y)$, then $a \oplus x = a \oplus y$. From $a \ominus x \leq x$, we have $x \leq a \oplus x$, then $x \leq a \oplus y$. It follows that $a \ominus x \leq y$, which implies $x \leq y$. Similarly, we can prove $y \leq x$ and hence $x = y$. Consequently, we obtain that u is injective.

(4) One can easily prove the fact that $x \vee y = x \oplus (y \ominus x) = y$ if and only if there exists $z \in L$ such that $y \ominus z = x$ for all $x, y \in L$. From the above fact, we will prove that u is surjective. Now, for all $x \in Fix_{g_a}(L)$, then $x = g_a(x) = a \oplus x$. Using the above fact, we have that $u(x \ominus a) = a \oplus (x \ominus a) = a \oplus ((a \oplus x) \ominus a) = a \oplus x = x$. Thus, we conclude that u is surjective. Also, it is easy to verify that $u^{-1}(x) = x \ominus a$ is a homomorphism from $Fix_{g_a}(L)$ to $Fix_{d_a^*}(L)$.

Combining them, we obtain that u is an isomorphism from $(Fix_{d_a^*}(L), \oplus, \neg_2, 0, a^*)$ to $(Fix_{g_a}(L), \oplus, \circ_1, a, 1)$. Therefore, MV-algebras $(Fix_{d_a^*}(L), \oplus, \neg_2, 0, a^*)$ and $(Fix_{g_a}(L), \oplus, \circ_1, a, 1)$ are isomorphic.

In the following, we prove a representation theorem for MV-algebras

Theorem 4.16. Let L be an MV-algebra and $a \in B(L)$. Then MV-algebra L is isomorphic to the direct product $(Fix_{d_a}(L), \oplus, \neg_1, 0, a)$ and $(Fix_{d_a^*}(L), \oplus, \neg_2, 0, a^*)$.

Proof. For all $a \in L$, let $\tau : L \longrightarrow Fix_{d_a}(L) \times Fix_{d_a^*}(L)$ be defined by $\tau(x) = (x \odot a, x \ominus a)$ for all $x \in L$. Clearly, τ is a map from L to $Fix_{d_a}(L) \times Fix_{d_a^*}(L)$, that is, τ is well defined.

(1) From Theorem 3.13 and 4.1, we get τ is a homomorphism from L to $Fix_{d_a}(L) \times Fix_{d_a^*}(L)$.

(2) Now, we prove that τ is injective. If $x_1 \in Fix_{d_a}(L)$ and $x_2 \in Fix_{d_a^*}(L)$, then for $x = x_1 \vee x_2$, we have $\tau(x) = (x_1, x_2)$ since (L, \wedge, \vee) is a distributive lattice and $x = (x \wedge a) \vee (x \wedge a^*)$ for all $x \in L$. Consequently, we obtain that τ is injective. Also, it is easy to verify that $\tau^{-1}(x, y) = x \vee y$ for all $(x, y) \in Fix_{d_a}(L) \times Fix_{d_a^*}(L)$ is a homomorphism from $Fix_{d_a}(L) \times Fix_{d_a^*}(L)$ to L .

Combining them, we obtain that τ is an isomorphism from L to the direct product $(Fix_{d_a}(L), \oplus, \neg_1, 0, a)$ and $(Fix_{d_a^*}(L), \oplus, \neg_2, 0, a^*)$. Therefore, MV-algebra L is isomorphic to the direct product $(Fix_{d_a}(L), \oplus, \neg_1, 0, a)$ and $(Fix_{d_a^*}(L), \oplus, \neg_2, 0, a^*)$.

Theorem 4.17. Let L be an MV-algebra and $a \in B(L)$. Then MV-algebra L is isomorphic to the direct product $(Fix_{g_a^*}(L), \oplus, \circ_2, a^*, 1)$ and $(Fix_{g_a}(L), \oplus, \circ_1, a, 1)$.

Proof. For all $a \in L$, let $\phi : L \longrightarrow Fix_{g_a}(L) \times Fix_{g_{a^*}}(L)$ be defined by $\phi(x) = (x \oplus a, x \rightarrow a)$ for all $x \in L$. Clearly, ϕ is a map from L to $Fix_{g_a}(L) \times Fix_{g_{a^*}}(L)$, that is, ϕ is well defined.

(1) From Theorems 4.11 and Corollary 4.13, we get ϕ is a homomorphism from L to $Fix_{g_a}(L) \times Fix_{g_{a^*}}(L)$.

(2) Now, we prove that ϕ is injective. If $x_1 \in Fix_{g_a}(L)$ and $x_2 \in Fix_{g_{a^*}}(L)$, then for $x = x_1 \wedge x_2$, we have $\phi(x) = (x_1, x_2)$ since (L, \wedge, \vee) is a distributive lattice and $x = (x \vee a) \wedge (x \vee a^*)$ for all $x \in L$. Consequently, we obtain that ϕ is injective. Also, it is easy to verify that $\phi^{-1}(x, y) = x \wedge y$ for all $(x, y) \in Fix_{g_a}(L) \times Fix_{g_{a^*}}(L)$ is a homomorphism from $Fix_{g_a}(L) \times Fix_{g_{a^*}}(L)$ to L .

Combining them, we obtain that ϕ is an isomorphism from L to the direct product $(Fix_{g_{a^*}}(L), \oplus, \circ_2, a^*, 1)$ and $(Fix_{g_a}(L), \oplus, \circ_1, a, 1)$. Therefore, MV-algebra L is isomorphic to the direct product $(Fix_{g_{a^*}}(L), \oplus, \circ_2, a^*, 1)$ and $(Fix_{g_a}(L), \oplus, \circ_1, a, 1)$.

From Theorem 4.14-4.17, the following theorems are immediate consequence.

Theorem 4.18. Let L be an MV-algebra and $a \in B(L)$. Then MV-algebra L is isomorphic to the direct product $(Fix_{d_a}(L), \oplus, \neg_1, 0, a)$ and $(Fix_{g_a}(L), \oplus, \circ_1, a, 1)$.

Proof. It follows from Theorem 4.14 and 4.16.

Theorem 4.19. Let L be an MV-algebra and $a \in B(L)$. Then MV-algebra L is isomorphic to the direct product $(Fix_{d_{a^*}}(L), \oplus, \neg_2, 0, a^*)$ and $(Fix_{g_{a^*}}(L), \oplus, \circ_2, a^*, 1)$.

Proof. It follows from Theorem 4.15 and 4.17.

In what follows, as applications of Boolean derivations, some characterizations of Boolean algebras will be given.

Theorem 4.20. Let L be an MV-algebra. Then the following statements are equivalent:

- (1) L is a Boolean algebra,
- (2) for any $a \in L$, $Fix_{d_a}(L) = (a]$,
- (3) for any $a \in L$, $Fix_{g_a}(L) = [a)$.

Proof. (1) \Rightarrow (2) Assume that L is a Boolean algebra, we have $x \odot x = x$ for all $x \in L$. It follows that $d_a(a) = a \odot a = a$ for all $a \in L$. Thus, $a \in Fix_{d_a}(L)$. Since L is a Boolean algebra, we have that $Fix_{d_a}(L)$ is an ideal of L by Proposition 3.19 and Theorem 4.1. That is, for all $x \in L$, if $x \leq a$, we obtain that $x \in Fix_{d_a}(L)$, which implies that $(a] \subseteq Fix_{d_a}(L)$. Next, we will show that $Fix_{d_a}(L) \subseteq (a]$. For all $x \in Fix_{d_a}(L)$, we have $d_a(x) = a \odot x = a \wedge x = x$, which implies that $x \leq a$, that is, $x \in (a]$. It follows that $Fix_{d_a}(L) \subseteq (a]$. Therefore, we obtain that $Fix_{d_a}(L) = (a]$.

(2) \Rightarrow (1) Assume that $Fix_{d_a}(L) = (a]$ for all $a \in L$. Since $a \in (a]$, we get $a \in Fix_{d_a}(L)$. It follows that $d_a(a) = a$, that is, $a \odot a = a$ for all $a \in L$. Therefore, we obtain that L is a Boolean algebra.

(1) \Rightarrow (3) Assume that L is a Boolean algebra, we have $x \oplus x = x$ for all $x \in L$. It follows that $g_a(a) = a \oplus a = a$ for all $a \in L$. Thus, $a \in Fix_{g_a}(L)$. Since L is a Boolean algebra, one can easily check that $Fix_{g_a}(L)$ is a filter of L by Theorem 4.5. That is, for all $x \in L$, if $a \leq x$, we obtain that $x \in Fix_{g_a}(L)$, which implies that $[a] \subseteq Fix_{g_a}(L)$. Next, we will show that $Fix_{g_a}(L) \subseteq [a]$. For all $x \in Fix_{g_a}(L)$, we have $g_a(x) = a \oplus x = a \vee x = x$, which implies that $a \leq x$, that is, $x \in [a]$. It follows that $Fix_{g_a}(L) \subseteq [a]$. Therefore, we obtain that $Fix_{g_a}(L) = [a]$.

(3) \Rightarrow (1) Assume that $Fix_{g_a}(L) = [a]$ for all $a \in L$. Since $a \in [a]$, we get $a \in Fix_{g_a}(L)$. It follows that $g_a(a) = a$, that is, $a \oplus a = a$ for all $a \in L$. Therefore, we obtain that L is a Boolean algebra.

In what follows, we focus on algebraic structure the set of all additive (implicative) Boolean derivations. We denote by $D(L) = \{d_a | a \in B(L)\}$ ($G(L) = \{g_a | a \in B(L)\}$) be the set of all additive (implicative) Boolean derivations of L .

Theorem 4.21. Let L be an MV-algebra. Then $(D(L), \sqcup, \sqcap, \star, d_0, d_1)$ is a Boolean algebra, where $(d_a \sqcup d_b)x = (d_a x) \vee (d_b x)$, $(d_a \sqcap d_b)x = (d_a x) \wedge (d_b x)$, $(d_a)^{\star}x = d_a^*x$, $(d_0)^{\star} = d_1$, for any $d_a, d_b \in D(L)$.

Proof. First, we show that $(D(L), \sqcup, \sqcap, d_0, d_1)$ is a bounded lattice with d_0 as the smallest element and d_1 as the greatest element. For all $d_a, d_b \in D(L)$ and $x \in L$, we have $(d_a \sqcap d_b)(x) = (d_a(x)) \wedge (d_b(x)) = (a \odot x) \wedge (b \odot x) = (a \wedge x) \wedge (b \wedge x) = (a \wedge b) \odot x = d_{a \wedge b}(x)$, that is, $(d_a \sqcap d_b) = d_{a \wedge b}$ and hence $d_a \sqcap d_b \in D(L)$. Furthermore, we have $(d_a \sqcup d_b)(x) = (d_a(x) \vee d_b(x)) = (a \odot x) \vee (b \odot x) = (a \wedge x) \vee (b \wedge x) = (a \vee b) \wedge x = (a \vee b) \odot x = d_{a \vee b}(x)$, that is, $(d_a \sqcup d_b) = d_{a \vee b}$ and hence $d_a \sqcup d_b \in D(L)$. Therefore, $(D(L), \sqcup, \sqcap, d_0, d_1)$ is a lattice. For all $d_a \in D(L)$ and $x \in L$, we have $(d_a \sqcap d_0)(x) = d_a(x) \wedge d_0(x) = 0 = d_0(x)$ and $(d_a \sqcup d_1)(x) = d_a(x) \vee d_1(x) = x = d_1(x)$. Thus, d_0 is the smallest element and d_1 is the greatest element in $D(L)$.

Next, we prove that $(D(L), \sqcup, \sqcap)$ is a distributive lattice. For all d_a, d_b, d_c and $x \in L$, from (L, \vee, \wedge) is a distributive lattice, one can prove $(d_a \sqcup (d_b \sqcap d_c)) = (d_a \sqcup d_b) \sqcap (d_a \sqcup d_c)$ and $(d_a \sqcap (d_b \sqcup d_c)) = (d_a \sqcap d_b) \sqcup (d_a \sqcap d_c)$. Therefore, $(D(L), \sqcup, \sqcap, d_0, d_1)$ is a bounded distributive lattice.

Finally, we prove that $(D(L), \sqcup, \sqcap, \star, d_0, d_1)$ is a Boolean algebra. For all $d_a \in D(L)$ and $x \in L$, we have $(d_a)^{\star}(x) = d_a^*(x) = a^* \odot x = d_{a^*}(x)$, that is, $(d_a)^{\star}(x) = d_{a^*}(x)$. One can easily check that if $a \in B(L)$, then $a^* \in B(L)$, and hence $d_{a^*} \in D(L)$. Furthermore, we have $(d_a \sqcup (d_a)^{\star})(x) = (d_a)(x) \vee d_{a^*}(x) = (a \odot x) \vee (a^* \odot x) = (a \vee a^*) \odot x = x = d_1(x)$ and $(d_a \sqcap (d_a)^{\star})(x) = (d_a)(x) \wedge d_{a^*}(x) = (a \odot x) \wedge (a^* \odot x) = (a \vee a^*) \wedge x = 0 = d_0(x)$, that is, $(d_a \sqcup (d_a)^{\star}) = d_1$ and $(d_a \sqcap (d_a)^{\star}) = d_0$. Therefore, $(D(L), \sqcup, \sqcap, \star, d_0, d_1)$ is a Boolean algebra.

Theorem 4.22. Let L be an MV-algebra. Then $(G(L), \cap, \cup, \bullet, g_0, g_1)$ is a Boolean algebra, where $(g_a \cup g_b)x = (g_a x) \vee (g_b x)$, $(g_a \cap g_b)x = (g_a x) \wedge (g_b x)$, $(g_a)^{\bullet}x = g_{a^*}x$, $(g_0)^{\bullet} = g_1$, for any $g_a, g_b \in G(L)$.

Proof. The proof is similar to that of Theorem 4.21.

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6 **Theorem 4.23.** Let L be an MV-algebra. Then Boolean algebras $(B(L), \wedge, \vee, *, 0, 1)$
7 and $(D(L), \sqcup, \sqcap, \star, d_0, d_1)$ are isomorphic.
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10 **Proof.** Let $\phi : B(L) \longrightarrow D(L)$ be defined by $\phi(a) = d_a$ for all $a \in B(L)$. Clearly,
11 ϕ is a map from $B(L)$ to $D(L)$, that is, ϕ is well defined. One can easily see that
12 ϕ is one to one and onto.
13

14 Furthermore, for any $a, b \in B(L)$, we have $\phi(a \wedge b) = d_{a \wedge b} = d_a \sqcap d_b = \phi(a) \sqcap \phi(b)$,
15 $\phi(a \vee b) = d_{a \vee b} = d_a \sqcup d_b = \phi(a) \sqcup \phi(b)$ and $\phi(a^*) = d_{a^*} = (d_a)^* = (\phi(a))^*$.
16

17 Combining them, we obtain that ϕ is an isomorphism from $(B(L), \wedge, \vee, *, 0, 1)$
18 to $(D(L), \sqcup, \sqcap, \star, d_0, d_1)$. Therefore, Boolean algebras $(B(L), \wedge, \vee, *, 0, 1)$ and $(D(L),$
19 $\sqcup, \sqcap, \star, d_0, d_1)$ are isomorphic.
20

21 **Theorem 4.24.** Let L be an MV-algebra. Then Boolean algebras $(B(L), \wedge, \vee, *, 0, 1)$
22 and $(G(L), \cap, \cup, \bullet, g_0, g_1)$ are isomorphic.
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25 **Proof.** The proof is similar to that of Theorem 4.23.
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28 The next theorem is now an immediate consequence.
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31 **Theorem 4.25.** Let L be a Boolean algebra. Then
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33 (1) $(L, \wedge, \vee, *, 0, 1)$ is isomorphic to $(D(L), \sqcup, \sqcap, \star, d_0, d_1)$,
34 (2) $(L, \wedge, \vee, *, 0, 1)$ is isomorphic to $(G(L), \cap, \cup, \bullet, g_0, g_1)$.
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37 **Proof.** It follows from Theorem 4.23, 4.24.
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5. Conclusions

40 The notion of derivations gives a tool for studying structures and properties in
41 algebraic systems. In the paper, some related properties of particular derivations
42 are discussed. Also, we obtain that the fixed point set of additive derivations is
43 still an MV-algebra. Besides, we get that the fixed point set of Boolean additive
44 derivations and that of their adjoint derivations are isomorphism. Finally, we obtain
45 that every MV-algebra is isomorphic to the direct product of the fixed point set of
46 Boolean additive derivations and that of their adjoint derivations. In the future,
47 we will consider the open problem in this paper.
48
49

Compliance with Ethical Standard

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54

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58 **Informed consent:** Informed consent was obtained from all individual participants
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60
61

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